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One-Parameter Irreducible Representation of spl(2, 1) Superalgebra

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Abstract One-parameter irreducible representation of the spl(2, 1) superalgebra is studied on subspace and quotient spaces of the universal enveloping algebra of Heisenberg-Weyl superalgebra. The parameter α may be related to the interaction parameter U in one exactly solvable model for correlated electrons.

Keywords spl(2, 1) superalgebra · Irreducible representation · Exactly solvable model

1 Introduction

A series of models of correlated electrons on a lattice and exactly solvable in one dimension and supersymmetric, such as Hubbard and extended Hubbard models and t-J model, EKS model, BGLZ model [1], has been extensively studied due to their promising role in theoretical condensed-matter physics and possibly in high-T_c superconductivity. Those models contain one symmetry-preserving free real parameter which is the Hubbard interaction parameter U. One-parameter irreducible representations of Lie superalgebra have played an important role in constructing supersymmetrical models. The supersymmetrical algebra of BGLZ model for correlated electrons on the unrestricted 4^L-dimensional electronic Hilbert space $\bigotimes_{n=1}^{L} C^4$ is superalgebra gl(2|1). Some irreducible representations of Lie superalgebras spl(2, 1) and gl(2|1) have been given by Chen [2–4]. One-parameter indecomposable representation of the spl(2, 1) superalgebra has been studied [5]. In the present paper we shall be concerned with the spl(2, 1) superalgebra. The purpose of the present paper is to study new one-parameter irreducible representation of the spl(2, 1) superalgebra on subspace and quotient spaces of the universal enveloping algebra of Heisenberg-Weyl superalgebra in terms of indecomposable representation of this superalgebra.

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2 One-Parameter Irreducible Representation of the spl(2, 1)

In accordance with Chen [2] the generators of the spl(2, 1) superalgebra read as follows:

$$\{Q_3, Q_+, Q_-, B \in \mathrm{spl}(2,1)_{\bar{0}} \mid V_+, V_-, W_+, W_- \in \mathrm{spl}(2,1)_{\bar{1}}\} \tag{1}$$

and satisfy the following commutation and anticommutation relations:

$$\begin{aligned}
&[Q_3, Q_\pm] = \pm Q_\pm, \qquad [Q_+, Q_-] = 2Q_3, \qquad [B, Q_\pm] = [B, Q_3] = 0\\
&[Q_3, V_\pm] = \pm\frac{1}{2}V_\pm, \qquad [Q_3, W_\pm] = \pm\frac{1}{2}W_\pm, \qquad [B, V_\pm,] = \frac{1}{2}V_\pm\\
&[B, W_\pm] = -\frac{1}{2}W_\pm, \qquad [Q_\pm, V_\mp] = V_\pm, \qquad [Q_\pm, W_\mp] = W_\pm, \qquad [Q_\pm, V_\pm] = 0\\
&[Q_\pm, W_\pm] = 0, \qquad \{V_\pm, V_\pm\} = \{V_\pm, V_\mp\} = \{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0,\\
&\{V_\pm, W_\pm\} = \pm Q_\pm, \qquad \{V_\pm, W_\mp\} = -Q_3 \pm B.
\end{aligned} \tag{2}$$

In Ref. [5] we have obtained one-parameter indecomposable representation of the spl(2, 1) superalgebra.

The generalized Fock space is defined as a quotient space of V

$$Y = (V/J) : \{\phi(k, \alpha_1, \alpha_2) = \phi(k, 0, \alpha_1, 0, \alpha_2, 0) \bmod J \mid k \in Z^+, \alpha_1, \alpha_2 = 0, 1\} \tag{3}$$

where J is the left ideal generated by the element $b - \lambda$, $a_1 - \eta_1$ and $a_2 - \eta_2$, λ is a complex number and η_1 and η_2 are generators of the Grassmann algebra $\tilde{G}$. On this space, a new representation form can be obtained as follows

$$\begin{aligned}
L(Q_3)\phi(k, \alpha_1, \alpha_2) &= \left(-\frac{1}{2}n + k + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2\right)\phi(k, \alpha_1, \alpha_2),\\
L(Q_+)\phi(k, \alpha_1, \alpha_2) &= (n - k - \alpha_1 - \alpha_2)\phi(k+1, \alpha_1, \alpha_2),\\
L(Q_-)\phi(k, \alpha_1, \alpha_2) &= k\phi(k-1, \alpha_1, \alpha_2),\\
L(B)\phi(k, \alpha_1, \alpha_2) &= \left[\left(\frac{1}{2} + \alpha\right)n - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2\right]\phi(k, \alpha_1, \alpha_2),\\
L(V_+)\phi(k, \alpha_1, \alpha_2) &= \alpha_1\sqrt{\alpha}\phi(k+1, \alpha_1 - 1, \alpha_2)\\
&\quad + (-1)^{\alpha_1}(1 - \alpha_2)(n - k - \alpha_1)\sqrt{1+\alpha}\phi(k, \alpha_1, \alpha_2 + 1),\\
L(V_-)\phi(k, \alpha_1, \alpha_2) &= \alpha_1\sqrt{\alpha}\phi(k, \alpha_1 - 1, \alpha_2)\\
&\quad - (-1)^{\alpha_1}(1 - \alpha_2)\sqrt{1+\alpha}k\phi(k-1, \alpha_1, \alpha_2 + 1),\\
L(W_+)\phi(k, \alpha_1, \alpha_2) &= (-1)^{\alpha_1}\alpha_2\sqrt{1+\alpha}\phi(k+1, \alpha_1, \alpha_2 - 1)\\
&\quad + (-n + k + \alpha_2)(1 - \alpha_1)\sqrt{\alpha}\phi(k, \alpha_1 + 1, \alpha_2),\\
L(W_-)\phi(k, \alpha_1, \alpha_2) &= (1 - \alpha_1)\sqrt{\alpha}k\phi(k-1, \alpha_1 + 1, \alpha_2)\\
&\quad + (-1)^{\alpha_1}\alpha_2\sqrt{1+\alpha}\phi(k, \alpha_1, \alpha_2 - 1).
\end{aligned} \tag{4}$$

We can easily see that the representation (4) is an infinite-dimensional irreducible representation when $n \notin Z^+$. Obviously, the invariant subspace exists when $n \in Z^+$,

$$\begin{aligned} &Y(n): \{\phi(k, \alpha_1, \alpha_2) \in Y \mid k + \alpha_1 + \alpha_2 \le n, k \in Z^+, \alpha_1, \alpha_2 = 0, 1\}, \\ &\dim Y(n) = 4n, \end{aligned} \tag{5}$$

and there is no invariant complementary subspace. Thus, the representation (4) is indecomposable. Restricting the representation given by (4) to the invariant subspace $Y(n)$, we can obtain a finite-dimensional irreducible representation of the spl(2, 1).

For the sake of simplicity, we redefine the basis of $Y(n)$ as

$$\begin{aligned} |j, m, \alpha_1, \alpha_2\rangle = &\sqrt{\frac{(j-m)!(2j-\alpha_1-\alpha_2)!}{(j+m)!(j-m-\alpha_1)!(j-m-\alpha_2)!}} \\ &\times \phi(j+m, \alpha_1, \alpha_2), \end{aligned} \tag{6}$$

where $j = \frac{1}{2}n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots,$

$$\begin{aligned} &m = -j, -j+1, \ldots, j, \quad \text{when } \alpha_1 = 0, \alpha_2 = 0, \\ &m = -j, -j+1, \ldots, j-1, \quad \text{when } \alpha_1 = 0, \alpha_2 = 1, \\ &m = -j, -j+1, \ldots, j-1, \quad \text{when } \alpha_1 = 1, \alpha_2 = 0, \\ &m = -j, -j+1, \ldots, j-2, \quad \text{when } \alpha_1 = 1, \alpha_2 = 1. \end{aligned} \tag{7}$$

The action of the generators of the spl(2, 1) on the new basis vector is straightforwardly obtained with the help of (4) and (6). One finds

$$\begin{aligned} &Q_3|j, m, \alpha_1, \alpha_2\rangle = \left(m + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2\right)|j, m, \alpha_1, \alpha_2\rangle, \\ &Q_+|j, m, \alpha_1, \alpha_2\rangle = (j-m-\alpha_1-\alpha_2)\sqrt{\frac{(j+m+1)(j-m)}{(j-m-\alpha_1)(j-m-\alpha_2)}}|j, m+1, \alpha_1, \alpha_2\rangle, \\ &Q_-|j, m, \alpha_1, \alpha_2\rangle = \sqrt{\frac{(j+m)(j-m+1-\alpha_1)(j-m+1-\alpha_2)}{j-m+1}}|j, m-1, \alpha_1, \alpha_2\rangle, \\ &B|j, m, \alpha_1, \alpha_2\rangle = \left[(1+2\alpha)j - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2\right]|j, m, \alpha_1, \alpha_2\rangle, \\ &V_+|j, m, \alpha_1, \alpha_2\rangle = (-1)^{\alpha_1}(1-\alpha_2)\sqrt{1+\alpha}(j-m-\alpha_1)\frac{1}{\sqrt{j-m-\alpha_2}}|j, m, \alpha_1, \alpha_2+1\rangle \\ &\qquad + \alpha_1\sqrt{\alpha}\sqrt{\frac{(j+m+1)(j-m)}{j-m-\alpha_2}}|j, m+1, \alpha_1-1, \alpha_2\rangle, \\ &V_-|j, m, \alpha_1, \alpha_2\rangle = \alpha_1\sqrt{\alpha}\sqrt{j-m+1-\alpha_1}|j, m, \alpha_1-1, \alpha_2\rangle \\ &\qquad - (-1)^{\alpha_1}(1-\alpha_2)\sqrt{1+\alpha}\sqrt{\frac{j+m}{(j-m+1)(j-m-\alpha_2)}} \\ &\qquad \times |j, m-1, \alpha_1, \alpha_2+1\rangle \end{aligned} \tag{8}$$

$$W_+|j,m,\alpha_1,\alpha_2\rangle = -(1-\alpha_1)\sqrt{\alpha}(j-m-\alpha_2)\frac{1}{\sqrt{j-m-\alpha_1}}|j,m,\alpha_1+1,\alpha_2\rangle + (-1)^{\alpha_1}\alpha_2\sqrt{1+\alpha}\sqrt{\frac{(j+m+1)(j-m)}{j-m-\alpha_1}}|j,m+1,\alpha_1,\alpha_2-1\rangle$$

$$W_-|j,m,\alpha_1,\alpha_2\rangle = (1-\alpha_1)\sqrt{\alpha}\sqrt{\frac{(j+m)(j-m+1-\alpha_2)}{j-m+1}}|j,m-1,\alpha_1+1,\alpha_2\rangle + (-1)^{\alpha_1}\alpha_2\sqrt{1+\alpha}\sqrt{j-m+1-\alpha_2}|j,m,\alpha_1,\alpha_2-1\rangle,$$

where we restrict $|j,j+1,\alpha_1,\alpha_2\rangle = |j,-j-1,\alpha_1,\alpha_2\rangle = 0$.

3 Conclusion

To illustrate the irreducibility of the spl(2, 1) representation, we have a simple discussion. In the first place, the representation space $Y(2j)$ set up by all $|j,m,\alpha_1,\alpha_2\rangle$ marked with j is invariant under the action of the spl(2,1) generators. In the next place, there is no true subspace in the $Y(2j)$. It is clear from (8) that this representation is a $8j$-dimensional irreducible representation.

We have obtained one-parameter irreducible representation. All the finite-dimensional one-parameter irreducible representation of the spl(2, 1) have been given on the subspace of the generalized Fock space. In terms of the conclusion it may be of use for further researches on one-parameter coherent state of the spl(2,1) superalgebra and for determining new supersymmetrical quantum model corresponding to the spl(2, 1).

Appendix

Here we derive the new representation form (4) in terms of one-parameter indecomposable representation of the spl(2, 1) superalgebra. In following space

$$V: \{\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)|k,l\in Z^+,\alpha_1,\beta_1,\alpha_2,\beta_2=0,1\} \tag{A.1}$$

its explicit form as follows:

$$\begin{aligned}
&L(Q_3)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad=\left(-\frac{1}{2}n+k+\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2\right)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)+\phi(k+1,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\qquad+\frac{1}{2}(-1)^{\alpha_1}(1-\alpha_1)\phi(k,l,\alpha_1+1,\beta_1+1,\alpha_2,\beta_2)\\
&\qquad+\frac{1}{2}(-1)^{\alpha_2}(1-\alpha_2)\phi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2+1),\\
&L(Q_+)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad=(n-k-\alpha_1-\alpha_2)\phi(k+1,l,\alpha_1,\beta_1,\alpha_2,\beta_2)-\phi(k+2,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\qquad-(-1)^{\alpha_1}(1-\alpha_1)\phi(k+1,l,\alpha_1+1,\beta_1+1,\alpha_2,\beta_2)
\end{aligned}$$

$$
\begin{aligned}
&- (-1)^{\alpha_2}(1-\alpha_2)\phi(k+1,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2+1),\\
&L(Q_-)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2) = \phi(k,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2) + k\phi(k-1,l,\alpha_1,\beta_1,\alpha_2,\beta_2),\\
&L(B)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad = \left[\left(\frac{1}{2}+\alpha\right)n - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2\right]\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\qquad + \frac{1}{2}(-1)^{\alpha_1}(1-\alpha_1)\phi(k,l,\alpha_1+1,\beta_1+1,\alpha_2,\beta_2)\\
&\qquad + \frac{1}{2}(-1)^{\alpha_2}(1-\alpha_2)\phi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2+1),\\
&L(V_+)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad = (-1)^{\alpha_1+\beta_1}(1-\alpha_2)(n-k-\alpha_1)\sqrt{1+\alpha}\phi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2)\\
&\qquad + (-1)^{\alpha_1}\sqrt{\alpha}\phi(k+1,l,\alpha_1,\beta_1+1,\alpha_2,\beta_2) \qquad\qquad \text{(A.2)}\\
&\qquad + \alpha_1\sqrt{\alpha}\phi(k+1,l,\alpha_1-1,\beta_1,\alpha_2,\beta_2)\\
&\qquad - (-1)^{\alpha_1+\beta_1}(1-\alpha_2)\sqrt{1+\alpha}\phi(k+1,l+1,\alpha_1,\beta_1,\alpha_2+1,\beta_2)\\
&\qquad - (-1)^{\beta_1}(1-\alpha_1)(1-\alpha_2)\sqrt{1+\alpha}\phi(k,l,\alpha_1+1,\beta_1+1,\alpha_2+1,\beta_2),\\
&L(V_-)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad = (-1)^{\alpha_1}\sqrt{\alpha}\phi(k,l,\alpha_1,\beta_1+1,\alpha_2,\beta_2) + \alpha_1\sqrt{\alpha}\phi(k,l,\alpha_1-1,\beta_1,\alpha_2,\beta_2)\\
&\qquad - (-1)^{\alpha_1+\beta_1}(1-\alpha_2)\sqrt{1+\alpha}\phi(k,l+1,\alpha_1,\beta_1,\alpha_2+1,\beta_2)\\
&\qquad - (-1)^{\alpha_1+\beta_1}(1-\alpha_2)\sqrt{1+\alpha}k\phi(k-1,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2),\\
&L(W_+)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad = (-n+k+\alpha_2)(1-\alpha_1)\sqrt{\alpha}\phi(k,l,\alpha_1+1,\beta_1,\alpha_2,\beta_2)\\
&\qquad + (-1)^{\alpha_1+\beta_1+\alpha_2}\sqrt{1+\alpha}\phi(k+1,l,\alpha_1,\beta_1,\alpha_2,\beta_2+1)\\
&\qquad + (-1)^{\alpha_1+\beta_1}\alpha_2\sqrt{1+\alpha}\phi(k+1,l,\alpha_1,\beta_1,\alpha_2-1,\beta_2)\\
&\qquad + (1-\alpha_1)\sqrt{\alpha}\phi(k+1,l+1,\alpha_1+1,\beta_1,\alpha_2,\beta_2)\\
&\qquad + (-1)^{\alpha_2}(1-\alpha_1)(1-\alpha_2)\sqrt{\alpha}\phi(k,l,\alpha_1+1,\beta_1,\alpha_2+1,\beta_2+1),\\
&L(W_-)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad = (-1)^{\alpha_1+\beta_1+\alpha_2}\sqrt{1+\alpha}\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2+1)\\
&\qquad + (-1)^{\alpha_1+\beta_1}\alpha_2\sqrt{1+\alpha}\phi(k,l,\alpha_1,\beta_1,\alpha_2-1,\beta_2)\\
&\qquad + (1-\alpha_1)\sqrt{\alpha}\phi(k,l+1,\alpha_1+1,\beta_1,\alpha_2,\beta_2)\\
&\qquad + (1-\alpha_1)\sqrt{\alpha}k\phi(k-1,l,\alpha_1+1,\beta_1,\alpha_2,\beta_2).
\end{aligned}
$$

The generalized Fock space is defined as a quotient space of V

$$
Y = (V/J) : \{\phi(k,\alpha_1,\alpha_2) = \phi(k,0,\alpha_1,0,\alpha_2,0) \bmod J \mid k \in Z^+, \alpha_1,\alpha_2 = 0,1\} \tag{A.3}
$$

where J is the left ideal generated by the element $b-\lambda$, $a_1-\eta_1$ and $a_2-\eta_2$, λ is a complex number and η_1 and η_2 are generators of the Grassmann algebra $\tilde{G}$. On this space, the representation (A.2) induces the new representation

$$
\begin{aligned}
&L(Q_3)\phi(k,\alpha_1,\alpha_2)\\
&\quad=\left(-\frac{1}{2}n+k+\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2\right)\phi(k,\alpha_1,\alpha_2)+\lambda\phi(k+1,\alpha_1,\alpha_2)\\
&\qquad+\frac{1}{2}(1-\alpha_1)\eta_1\phi(k,\alpha_1+1,\alpha_2)+\frac{1}{2}(1-\alpha_2)\eta_2\phi(k,\alpha_1,\alpha_2+1),\\
&L(Q_+)\phi(k,\alpha_1,\alpha_2)\\
&\quad=(n-k-\alpha_1-\alpha_2)\phi(k+1,\alpha_1,\alpha_2)\\
&\qquad-\lambda\phi(k+2,\alpha_1,\alpha_2)-(1-\alpha_1)\eta_1\phi(k+1,\alpha_1+1,\alpha_2)\\
&\qquad-(1-\alpha_2)\eta_2\phi(k+1,\alpha_1,\alpha_2+1),\\
&L(Q_-)\phi(k,\alpha_1,\alpha_2)=\lambda\phi(k,\alpha_1,\alpha_2)+k\phi(k-1,\alpha_1,\alpha_2),\\
&L(B)\phi(k,\alpha_1,\alpha_2)\\
&\quad=\left[\left(\frac{1}{2}+\alpha\right)n-\frac{1}{2}\alpha_1-\frac{1}{2}\alpha_2\right]\phi(k,\alpha_1,\alpha_2)-\frac{1}{2}(1-\alpha_1)\eta_1\phi(k,\alpha_1+1,\alpha_2)\\
&\qquad-\frac{1}{2}(1-\alpha_2)\eta_2\phi(k,\alpha_1,\alpha_2+1),\\
&L(V_+)\phi(k,\alpha_1,\alpha_2)\\
&\quad=(-1)^{\alpha_1}\sqrt{\alpha}\eta_1\phi(k+1,\alpha_1,\alpha_2)\\
&\qquad+(-1)^{\alpha_1}(1-\alpha_2)(n-k-\alpha_1)\sqrt{1+\alpha}\phi(k,\alpha_1,\alpha_2+1)\\
&\qquad-(-1)^{\alpha_1}(1-\alpha_2)\sqrt{1+\alpha}\lambda\phi(k+1,\alpha_1,\alpha_2+1)\\
&\qquad-(1-\alpha_1)(1-\alpha_2)\sqrt{1+\alpha}\eta_1\phi(k,\alpha_1+1,\alpha_2+1)\\
&\qquad+\alpha_1\sqrt{\alpha}\phi(k+1,\alpha_1-1,\alpha_2),\\
&L(V_-)\phi(k,\alpha_1,\alpha_2)\\
&\quad=(-1)^{\alpha_1}\sqrt{\alpha}\eta_1\phi(k,\alpha_1,\alpha_2)-(-1)^{\alpha_1}(1-\alpha_2)\sqrt{1+\alpha}\lambda\phi(k,\alpha_1,\alpha_2+1)\\
&\qquad-(-1)^{\alpha_1}(1-\alpha_2)\sqrt{1+\alpha}k\phi(k-1,\alpha_1,\alpha_2+1)\\
&\qquad+\alpha_1\sqrt{\alpha}\phi(k,\alpha_1-1,\alpha_2),\\
&L(W_+)\phi(k,\alpha_1,\alpha_2)\\
&\quad=(-1)^{\alpha_1}\alpha_2\sqrt{1+\alpha}\phi(k+1,\alpha_1,\alpha_2-1)\\
&\qquad+(-n+k+\alpha_2)(1-\alpha_1)\sqrt{\alpha}\phi(k,\alpha_1+1,\alpha_2)\\
&\qquad+(-1)^{\alpha_1+\alpha_2}\sqrt{1+\alpha}\eta_2\phi(k+1,\alpha_1,\alpha_2)\\
&\qquad+(1-\alpha_1)\sqrt{\alpha}\lambda\phi(k+1,\alpha_1+1,\alpha_2)+(1-\alpha_1)(1-\alpha_2)\sqrt{\alpha}\eta_2\phi(k,\alpha_1+1,\alpha_2+1),\\
&L(W_-)\phi(k,\alpha_1,\alpha_2)\\
&\quad=(-1)^{\alpha_1+\alpha_2}\sqrt{1+\alpha}\eta_2\phi(k,\alpha_1,\alpha_2)+(-1)^{\alpha_1}\alpha_2\sqrt{1+\alpha}\phi(k,\alpha_1,\alpha_2-1)\\
&\qquad+(1-\alpha_1)\sqrt{\alpha}\lambda\phi(k,\alpha_1+1,\alpha_2)+(1-\alpha_1)\sqrt{\alpha}k\phi(k-1,\alpha_1+1,\alpha_2).
\end{aligned}
\tag{A.4}
$$

The representation given by (A.4) is an infinite-dimensional irreducible representation for the cases $\lambda \neq 0, \eta_1 \neq 0$ or $\eta_2 \neq 0$. When $\lambda = \eta_1 = \eta_2 = 0$, the representation (A.4) becomes

$$\begin{aligned}
&L(Q_3)\phi(k,\alpha_1,\alpha_2) = \left(-\frac{1}{2}n + k + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2\right)\phi(k,\alpha_1,\alpha_2),\\
&L(Q_+)\phi(k,\alpha_1,\alpha_2) = (n-k-\alpha_1-\alpha_2)\phi(k+1,\alpha_1,\alpha_2),\\
&L(Q_-)\phi(k,\alpha_1,\alpha_2) = k\phi(k-1,\alpha_1,\alpha_2),\\
&L(B)\phi(k,\alpha_1,\alpha_2) = \left[\left(\frac{1}{2}+\alpha\right)n - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2\right]\phi(k,\alpha_1,\alpha_2),\\
&L(V_+)\phi(k,\alpha_1,\alpha_2)\\
&\quad = \alpha_1\sqrt{\alpha}\phi(k+1,\alpha_1-1,\alpha_2)\\
&\qquad + (-1)^{\alpha_1}(1-\alpha_2)(n-k-\alpha_1)\sqrt{1+\alpha}\phi(k,\alpha_1,\alpha_2+1),\\
&L(V_-)\phi(k,\alpha_1,\alpha_2)\\
&\quad = \alpha_1\sqrt{\alpha}\phi(k,\alpha_1-1,\alpha_2)\\
&\qquad - (-1)^{\alpha_1}(1-\alpha_2)\sqrt{1+\alpha}k\phi(k-1,\alpha_1,\alpha_2+1),\\
&L(W_+)\phi(k,\alpha_1,\alpha_2)\\
&\quad = (-1)^{\alpha_1}\alpha_2\sqrt{1+\alpha}\phi(k+1,\alpha_1,\alpha_2-1)\\
&\qquad + (-n+k+\alpha_2)(1-\alpha_1)\sqrt{\alpha}\phi(k,\alpha_1+1,\alpha_2),\\
&L(W_-)\phi(k,\alpha_1,\alpha_2)\\
&\quad = (1-\alpha_1)\sqrt{\alpha}k\phi(k-1,\alpha_1+1,\alpha_2) + (-1)^{\alpha_1}\alpha_2\sqrt{1+\alpha}\phi(k,\alpha_1,\alpha_2-1).
\end{aligned} \tag{A.5}$$

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